Exam - Statistics 2019/2020

Date: November 4, 2019
Time: 15.00-18.00
Place: MartiniPlaza (Springerlaan 2, 9727 KB Groningen)
Progress code: WISTAT-07

Rules to follow:

- This is a closed book exam. Consultation of books and notes is **not** permitted. You can use a simple (non-programmable) calculator.
- Do not forget to write your name and student number onto each paper sheet.
- There are 5 exercises and the number of points per exercise are indicated within boxes. You can reach 90 points.
- For derivations include the relevant equation(s) and/or a short description.
- **We wish you success with the completion of the exam!**

START OF EXAM

1. **Method of Moments estimator for Gamma distribution.** $\boxed{5}$
Given a random sample $X_1, \ldots, X_n$ from a Gamma distribution, derive the Methods of Moments (MOM) estimators of the two parameters.
If a random variable X has a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$, then its expectation and its variance are: $E[X] = \alpha/\beta$ and Var(X)= α/β^2.

2. **Random sample from Pareto distribution.** $\boxed{20}$
We consider a random sample $X_1, \ldots, X_n$ from a Pareto distribution. The density (pdf) of a Pareto distribution with parameters $\theta > 0$ and $a > 0$ is:

$$f_{\theta,a}(x) = \begin{cases} \theta \cdot a^{\theta} \cdot x^{-\theta-1}, & x \geq a \\ 0, & x < a \end{cases}$$

HINT: For ML estimators: If applicable, check via the 2nd derivative if they are really maximum points.

(a) $\boxed{5}$ Assume that the parameter a is known. Find a sufficient statistic for θ.

(b) $\boxed{5}$ Assume that the parameter θ is known. Find a sufficient statistic for a.

(c) $\boxed{5}$ Assume that the parameter a is known. Derive the ML estimator for θ.

(d) $\boxed{5}$ Assume that the parameter θ is known. Determine the ML estimator for a.

HINT FOR EXERCISES 4-5 (SEE NEXT PAGE):
$N(0,1)$ quantiles are: $q_{0.90} \approx 1.3$, $q_{0.95} \approx 1.6$, $q_{0.975} \approx 2.0$, and $q_{0.99} \approx 2.3$.

3. **Cramer-Rao bound.** $\boxed{\mathbf{20}}$
Consider a random sample $X_1, \dots, X_n$ from a $N(\mu, \sigma^2)$ distribution with known mean $\mu = 0$ and unknown variance $\sigma^2 > 0$. Recall the density (pdf) of a $N(\mu, \sigma^2)$:

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\} \qquad (x \in \mathbb{R})$$

   (a) $\boxed{5}$ Show that the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ is unbiased.

   (b) $\boxed{10}$ Show that the expected Fisher information (for sample size 1) is $(2\sigma^4)^{-1}$.

   (c) $\boxed{5}$ Check whether the estimator from (a) attains the Cramer-Rao bound.

HINT: Given a random sample $X_1, \dots, X_n$ from a $N(0, 1)$ distribution, it follows: $Z := \left(\sum_{i=1}^{n} X_i^2\right) \sim \chi_n^2$, so that $E[Z] = n$ and $Var(Z) = 2n$.

4. **Neyman Pearson (UMP test).** $\boxed{\mathbf{20}}$
Consider a random sample $X_1, \dots, X_9$ from a $N(\mu, \sigma^2)$ distribution with known variance $\sigma^2 = 1$ and unknown mean μ. Give the uniform most powerful (UMP) test to the level $\alpha = 0.1$ for the test problem $H_0 : \mu = 0$ vs. $H_1 : \mu = 1$, and check if the power of the test is greater than 0.95. Recall the density (pdf) of a $N(\mu, \sigma^2)$:

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\} \qquad (x \in \mathbb{R})$$

5. **Asymptotic confidence intervals and tests.** $\boxed{\mathbf{25}}$
Consider a random sample $X_1, \dots, X_n$ from a Negative Binomial distribution with known parameter $r \in \mathbb{N}$ and unknown probability parameter $\theta \in (0, 1)$. Recall that the density (pdf) and the expectation of a Negative Binomial are

$$f(x) = \binom{x+r-1}{x} \cdot (1-\theta)^r \cdot \theta^x \quad (x \in \mathbb{N}_0), \qquad E[X] = \frac{\theta r}{1-\theta}$$

   (a) $\boxed{5}$ Show that the ML estimator of θ is: $\hat{\theta}_{ML} = \bar{X}/(r + \bar{X})$.
   Don't forget to check via the 2nd derivative if this is really a maximum point.

   (b) $\boxed{5}$ Show that the expected Fisher information (for sample size 1) is

$$I(\theta) = \frac{r}{\theta \cdot (1-\theta)^2}$$

From now on we assume that $r = 2$, $n = 20$ and that $\bar{X} = 8$ has been observed.

   (c) $\boxed{5}$ Make use of the asymptotic normality of the ML estimator and give a two-sided asymptotic 95% confidence interval $[L, U]$ for θ.

   (d) $\boxed{5}$ Make use of the asymptotic normality of the ML estimator and give a one-sided asymptotic 95% confidence interval $(-\infty, U]$ for θ.

   (e) $\boxed{5}$ Check whether a score-test to the level $\alpha = 0.02$ would reject the null hypothesis $H_0 : \theta = 0.9$ in favour of the alternative $H_1 : \theta \neq 0.9$.
   HINT: Score test: $\frac{d}{d\theta} l_X(\theta) / \sqrt{n \cdot I(\theta)}$ is asymptotically $N(0, 1)$ distributed.

END OF EXAM